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LETTER TO THE EDITOR

q-oscillator representations of Hermitian braided matrices

M Arik and A Yildiz
Boğaziçi University, Center for Turkish–Balkan Physics Research and Applications, Bebek, Istanbul, Turkey

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Abstract. Representations of the braided algebra of $BM_q(2)$ in a Hilbert space are constructed. The generators of this algebra are expressed in terms of two independent q-oscillators. It is shown that, apart from a factor of $U(1)$, these generators reduce to the ordinary angular momentum algebra generators in the $q \to 1$ limit.

The role of Lie groups in physics and especially in quantum theory is crucial. Quantization of classical completely integrable nonlinear systems requires that the classical Lie group should also be quantized. Then one obtains a completely solvable quantum system. This has introduced the concept of quantum groups [1] into physics and mathematics. For a quantum matrix group the elements of a matrix do not commute between themselves. However, when the 'group product' is taken, elements of different matrices commute. Majid has argued that this property which already does not hold for supermatrices belonging to a supergroup, has to be generalized to define the so-called braided matrix group [2]. Mathematically speaking, a quantum matrix group is a Hopf algebra in the category of vector spaces with the usual commutative tensor product, whereas a braided matrix group is the same in a quasitensor category [3]. Braided groups are like supergroups with the property that the (± 1) superstatistics is replaced by $(-1 \leqslant q \leqslant 1)$ braid statistics. One of the main motivations of braided groups is the existence of particles with braid statistics in low-dimensional quantum field theories [4]. Braided groups can be used as a tool for performing quantum group calculations in a fully covariant way [2]. This is called the transmutation [5] of a quantum group into a braided group. After transmutation, the resulting braided group is braided commutative. There is also an adjoint process that converts any braided group into an ordinary quantum group [6]. So, braided geometry and super geometry are much closer to classical geometry than the quantum case. Since every quantum group can be viewed as a braided group we can conclude that braided group theory contains quantum group theory and supersymmetry. Majid showed [7] that the braided matrix group $BM_q(2)$ is isomorphic to the Sklyanin algebra [8] and the algebra generated by the elements of a Hermitian matrix which belongs to $BM_q(2)$ can be constructed in terms of the the generators of $U_q(su(2))$. In this letter we will construct the representations of this algebra by taking the generators as operators in a Hilbert space. We will present finite- and infinite-dimensional representations of these operators and we will show that the spectrum of a non-negative operator is simply the spectrum of a quadratic oscillator [9] described by the property that the commutation relation between creation and annihilation operators $a^\dagger$ and a can be expressed solely in terms of a quadratic function of $aa^\dagger$ and $a^\dagger a$. We define the Casimir of this algebra in terms of the quantum trace and the quantum determinant of the braided matrix.

The matrix elements of the quantum matrix group $SU_q(2)$ can be constructed in terms of one q-oscillator and one central unitary operator, which for irreducible representations in a Hilbert space is given by a complex phase [10]. On the other hand, Biedenharn and Macfarlane have constructed the quantum enveloping algebra $U_q(su(2))$ in terms of two independent q-deformed harmonic oscillators [11]. This generalizes the Schwinger construction used in the quantum theory of $(su(2))$ angular momentum. Using a similar approach we construct the algebra of Hermitian $BM_q(2)$ in terms of two independent q-deformed harmonic oscillators. We find that if the deformation parameter of $BM_q(2)$ is q, then one of the oscillators is a q-oscillator and the other is a q^{-1}-oscillator. The Fock space representations of these oscillators are constructed. Finally, we discuss the $q \to 1$ limit. If this limit is taken in a certain way we show that apart from a constant operator whose limit is singular, elements of $BM_q(2)$ reduce to the generators of $su(2)$.

An element $u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of the braided quantum matrix $BM_q(2)$ satisfies the relation

$$R_{21}u_1R_{12}u_2 = u_2R_{21}u_1R_{12} \tag{1}$$

where $u_1 = u \otimes I,\ u_2 = I \otimes u$ and the R-matrices are the $SL_q(2)$ R-matrices satisfying the quantum Yang–Baxter equation (QYBE). Equation (1) gives the relations

$$\begin{aligned} ba &= q^2 ab \\ ca &= q^{-2} ac \\ ad &= da \\ bc &= cb + (1-q^{-2})a(d-a) \\ db &= bd + (1-q^{-2})ab \\ cd &= dc + (1-q^{-2})ca. \end{aligned} \tag{2}$$

To find a representation in a Hilbert space we need to specify hermiticity conditions for a, b, c and d. To this end we choose u to be Hermitian and obtain $a^\dagger = a,\ d^\dagger = d$ and $c = b^\dagger$. It can be shown that $b^\dagger b,\ bb^\dagger,\ a$, and d form a commuting set. So for a representation in a Hilbert space we can take this set to be diagonal operators together with $b^\dagger$ and b acting as raising and lowering operators, respectively. In other words, we have

$$\begin{aligned} a\,|n\rangle &= a_n\,|n\rangle \\ d\,|n\rangle &= d_n\,|n\rangle \\ b\,|n\rangle &= b_n\,|n-1\rangle \\ b^\dagger\,|n\rangle &= b^*_{n+1}\,|n+1\rangle. \end{aligned} \tag{3}$$

Using the braided algebra relations (2) we find that these are satisfied provided that

$$\begin{aligned} a_{n+1} &= q^2 a_n \\ d_{n+1} &= d_n - (1-q^{-2})a_n \\ b^*_{n+1}b_{n+1} &= b^*_n b_n + (1-q^{-2})a_n(d_n - a_n). \end{aligned} \tag{4}$$

These difference equations have the solutions

$$\begin{aligned} a_n &= q^{2n}a_0 \\ d_n &= d_0 + a_0 q^{-2}(1-q^{2n}) \\ b^*_n b_n &= Aq^{4n} + Bq^{2n} + C \end{aligned} \tag{5}$$

where

$$\begin{aligned} A &= -a_0^2 q^{-4} \\ B &= a_0 q^{-2}(d_0 + q^{-2} a_0) \\ C &= b_0^* b_0 - q^{-2} a_0 d_0. \end{aligned} \tag{6}$$

The spectrum of $b^\dagger b$ is the same as the spectrum of the quadratic oscillator in [9]. To have the Fock space representations we should have a ground state which is annihilated by the lowering operator, i.e. $b|\ \rangle = 0$. Using equations (5) we find that the quantum determinant $ad - q^2 cb$ has the eigenvalue $a_0 d_0$ in this space and different values of d_0 will give different representations. Here a_0 and d_0 are integration constants of the difference equations. Together with the conditions $q^2 < 1$ and $a_0 d_0 < 0$, we have the infinite-dimensional representations

$$\left.\begin{aligned} a_n &= q^{2n} a_0 \\ d_n &= d_0 + a_0 q^{-2}(1 - q^{2n}) \\ b_n^* b_n &= a_0 q^{-2}(1 - q^{2n})(q^{2(n-1)} a_0 - d_0) \end{aligned}\right\} \qquad n = 0, 1, 2, \ldots . \tag{7}$$

For $d_0 = q^{2(N-1)} a_0$ we have the finite (N)-dimensional representation with states $|n\rangle = |0\rangle, |1\rangle, \ldots, |N-1\rangle$. The eigenvalues of the diagonal operators are

$$\begin{aligned} a_n &= q^{2n} a_0 \\ d_n &= a_0 q^{-2}(q^{2N} - q^{2n} + 1) \\ b_n^* b_n &= a_0^2 q^{-4}(1 - q^{2n})(q^{2n} - q^{2N}). \end{aligned} \tag{8}$$

Note that a_0 appears as a multiplicative factor for all elements ($a, b, b^\dagger$ and d) of matrix u and hence defines a commutative factor $\Re$. Taking the quotient of the Hermitian braided matrix by $\Re$ corresponds to choosing a particular value for a_0. This choice does not have to be a real number since a_0 can depend arbitrarily on any central element of the algebra generated by $a, b, b^\dagger$ and d. We define m, j and for later convenience choose a_0 such that

$$\begin{aligned} m &\equiv n - \frac{N-1}{2} \\ j &\equiv \frac{N-1}{2} \\ a_0 &= \frac{q^2}{1-q^2} A. \end{aligned} \tag{9}$$

Then equations (8) become

$$\begin{aligned} a_n &= \frac{q^{2(j+m+1)}}{1-q^2} A \\ d_n &= \frac{q^{2(2j+1)} - q^{2(j+m)} + 1}{1-q^2} A \\ b_n &= q^{j+m} \left(\frac{1 - q^{2(j+m)}}{1-q^2}\right)^{1/2} \left(\frac{1 - q^{2(j-m+1)}}{1-q^2}\right)^{1/2} A \end{aligned} \tag{10}$$

and the Casimir of the algebra is

$$C = \frac{q}{A(1-q^2)}(q^{-1} a + q d) - \frac{1+q^2}{q^4 A^2}(ad - q^2 cb). \tag{11}$$

In fact $q^{-1}a + qd$ is the quantum trace and $ad - q^2cb$ is simply the quantum determinant of braided matrices.

Consider two mutually commuting oscillators and number operators acting on Hilbert spaces such that

$$\begin{aligned}
&a_1 a_1^* - q a_1^* a_1 = 1\\
&a_2 a_2^* - q^2 a_2^* a_2 = q^{N_2+1}\\
&\left[a_{1,} a_2\right] = \left[a_1, a_2^*\right] = 0\\
&N_1 \left|n_1\right\rangle = n_1 \left|n_1\right\rangle\\
&N_2 \left|n_2\right\rangle = n_2 \left|n_2\right\rangle
\end{aligned} \tag{12}$$

where the a_i are lowering and the $a_i^\dagger$ are raising operators which give

$$\begin{aligned}
&a_1 \left|n_1\right\rangle = \left(\frac{1-q^{n_1}}{1-q}\right)^{1/2} \left|n_1 - 1\right\rangle\\
&a_1^* \left|n_1\right\rangle = \left(\frac{1-q^{n_1+1}}{1-q}\right)^{1/2} \left|n_1 + 1\right\rangle\\
&a_2 \left|n_2\right\rangle = \left(\frac{q^{n_2}\left(1-q^{n_2}\right)}{1-q}\right)^{1/2} \left|n_2 - 1\right\rangle\\
&a_2^* \left|n_2\right\rangle = \left(\frac{q^{n_2+1}\left(1-q^{n_2+1}\right)}{1-q}\right)^{1/2} \left|n_2 + 1\right\rangle
\end{aligned} \tag{13}$$

and by defining

$$\begin{aligned}
&j \equiv \frac{n_1+n_2}{2}\\
&m \equiv \frac{n_2-n_1}{2}
\end{aligned} \tag{14}$$

we obtain

$$\begin{aligned}
&a_1^* a_2 \left|n_1\right\rangle \left|n_2\right\rangle = \left(q^{j+m} \frac{\left(1-q^{j+m}\right)}{1-q} \frac{\left(1-q^{j-m+1}\right)}{1-q}\right)^{1/2} \left|j,\ m-1\right\rangle\\
&a_2^* a_1 \left|n_1\right\rangle \left|n_2\right\rangle = \left(q^{j+m+1} \frac{\left(1-q^{j+m+1}\right)}{1-q} \frac{\left(1-q^{j-m}\right)}{1-q}\right)^{1/2} \left|j,\ m+1\right\rangle\\
&\left(\frac{N_1+N_2}{2}\right) \left|n_1\right\rangle \left|n_2\right\rangle = j \left|j,\ m\right\rangle\\
&\left(\frac{N_2-N_1}{2}\right) \left|n_1\right\rangle \left|n_2\right\rangle = m \left|j,\ m\right\rangle.
\end{aligned} \tag{15}$$

The Casimir of the algebra formed by $a_1^* a_2,\ a_2^* a_1,\ a_1^* a_1$, etc, is $C' = N_1 + N_2$. By identifying

$$
\begin{aligned}
b &\equiv a_1^* a_2 A \\
b^* &\equiv a_2^* a_1 A \\
a &\equiv \frac{q^{N_2+1}}{1-q} A \\
d &\equiv \frac{q^{N_1+N_2+1} - q^{N_2} + 1}{1-q} A
\end{aligned}
\tag{16}
$$

and substituting q^2 instead of q, we see that this is simply a representation of the Hermitian braided algebra in $n_1 + n_2 + 1 = N$ dimensions. Therefore

$$
u = A \begin{pmatrix} \dfrac{q^{2(N_2+1)}}{1-q^2} & a_1^* a_2 \\ a_2^* a_1 & \dfrac{\left(q^{2(N_1+N_2+1)} - q^{2N_2} + 1\right)}{1-q^2} \end{pmatrix}
\tag{17}
$$

is a realization of the braided matrices $BM_q(2)$. Here $a_1^\dagger$ and a_1 form a linear q-oscillator [12], whereas $a_2^\dagger$ and a_2 form a squared q-oscillator [9].

The arbitrariness of the constant a_0 in (8) comes from the fact that when we multiply a braided group element by central elements the braided algebra relations remain invariant. Therefore we can define $A' = q^{(N_1+N_2)} A$. By using (12) and defining

$$
\begin{aligned}
c_1 &\equiv q^{-N_1} a_1 \\
c_1^* &\equiv q^{-N_1+1} a_1^* \\
c_2 &\equiv q^{-(N_2+1)} a_2 \\
c_2^* &\equiv q^{-N_2} a_2^*
\end{aligned}
\tag{18}
$$

we obtain

$$
\begin{aligned}
c_1^* c_2 &= q^{-(N_1+N_2)} a_1^* a_2 \\
c_1^* c_1 &= \frac{1-q^{-2N_1}}{1-q^{-2}} \\
c_2^* c_1 &= q^{-(N_1+N_2)} a_2^* a_1 \\
c_2^* c_2 &= \frac{1-q^{2N_2}}{1-q^2}
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
c_1 c_1^* - q^{-2} c_1^* c_1 &= 1 \\
c_2 c_2^* - q^{2} c_2^* c_2 &= 1.
\end{aligned}
\tag{20}
$$

Note the $q \to q^{-1}$ symmetry between these two independent linear q-oscillators. The matrix

$$
u = A' \begin{pmatrix} \dfrac{q^{N_2-N_1+2}}{1-q^2} & c_1^* c_2 \\ c_2^* c_1 & \dfrac{q^{N_1+N_2+2} - q^{N_2-N_1} + q^{-(N_1+N_2)}}{1-q^2} \end{pmatrix}
\tag{21}
$$

constructed in terms of these oscillators is a Hermitian braided matrix $BM_q(2)$. The Casimir of the braided algebra defined in (11) applied on a state $|j,\ m\rangle$ gives

$$C\,|j,\ m\rangle = q^2 \frac{(q^{2(j+1)} + q^{-2j} - 1 - q^2)}{(1-q^2)^2}\,|j,\ m\rangle. \tag{22}$$

We renormalize u by dividing out the central element A' and subtracting a q dependent multiple of the identity matrix. We then take the $q \to 1$ limit to obtain

$$\lim_{q\to 1}\left(\frac{u}{A'} - \frac{q^2}{1-q^2}I\right) = \begin{pmatrix} -\frac{(N_2-N_1)}{2} & a_1^* a_2 \\ a_2^* a_1 & \frac{(N_2-N_1)}{2} \end{pmatrix} \tag{23}$$

where a_1^*, a_1 a_2^* and a_2 are simply the creation and annihilation operators of two independent harmonic oscillators and N_1, N_2 are the corresponding number operators. Using the commutation relations between these operators we can identify

$$\begin{aligned} &\tfrac{1}{2}\,(N_2 - N_1) = J_3 \\ &a_1^* a_2 = J_- \\ &a_2^* a_1 = J_+ \end{aligned} \tag{24}$$

which satisfy the angular momentum algebra relations

$$\begin{aligned} &\left[J_3,\ J_\pm\right] = J_\pm \\ &\left[J_+,\ J_-\right] = 2J_3. \end{aligned} \tag{25}$$

This is simply the Schwinger construction of $su(2)$. The Casimir of the braided algebra (11) in the $q \to 1$ limit reduces to

$$\lim_{q\to 1} C = J_+ J_- + J_3^2 - J_3 \tag{26}$$

and the eigenvalue in $|j,\ m\rangle$ basis is found to be

$$C\,|j,\ m\rangle = j(j+1)\,|j,\ m\rangle. \tag{27}$$

This shows that finite-dimensional representations of the Hermitian $BM_q(2)/\Re$ constructed in this letter are in one-to-one correspondence with the finite-dimensional representations of $su(2)$. The unitarity condition $U^\dagger U = UU^\dagger = 1$ for matrices belonging to $BM_q(2)$ can be shown to be inconsistent with the commutations relations for the matrix elements of $BM_q(2)$. In contrast, the quantum group $SU_q(2)$ which can be obtained by imposing the unitarity condition on $SL_q(2)$ contains a single q-oscillator and the algebra satisfied by the matrix elements does not possess any finite-dimensional representations. On the other hand, from this point of view, the braided matrices $BM_q(2)$ with non-zero quantum determinant is similar to the q deformation $su_q(2)$ of the $su(2)$ Lie algebra.

In the Biedenharn–Macfarlane construction of $su_q(2)$ the generators J_+, J_- and J_3 are constructed in terms of two oscillators. However, these do not form a matrix. Another important difference between the Biedenharn–Macfarlane construction and ours is that in the former the two oscillators are identical copies of each other and there is a $q \to q^{-1}$ symmetry in normalization factors. In our construction the $q \to q^{-1}$ symmetry is preserved as the interchange symmetry of the two oscillators, one of which is a q-oscillator and the other is a q^{-1}-oscillator. Imposing the $q \to q^{-1}$ symmetry on the oscillators makes the construction unique among other choices. We would also like to mention that in our construction the oscillators are not braided in the sense introduced by Majid and Baskerville [13] in the context of the braided Heisenberg group.

Any one-dimensional generalized oscillator can be constructed in terms of another and in fact all one-dimensional oscillators can be constructed in terms of simple harmonic oscillator creation, annihilation and number operators. Since $BM_q(2)$ can be constructed in terms of the $su_q(2)$ algebra, and $su_q(2)$ can be constructed in terms of Biedenharn–Macfarlane oscillators, $BM_q(2)$ can also be constructed in terms of Biedenharn–Macfarlane oscillators. However, this would be a very complicated construction. The point of this letter is that the simplest construction of $BM_q(2)$ is in terms of a q, q^{-1} pair of simple q-oscillators.

The case of the quantum determinant being equal to zero ($ad - q^2cb = 0$), like the quantum group $SU_q(2)$, contains just one oscillator and does not have any finite-dimensional representations. To construct this representation we use (5) and set the quantum determinant to zero, and obtain the condition $d_0 = 0$. Finally, we obtain the result that the matrix

$$u = a_0' \begin{pmatrix} \dfrac{q^{2N}}{(q^{-2}-1)^{1/2}} & b \\ b^\dagger & \dfrac{q^{-2}(1-q^{2N})}{(q^{-2}-1)^{1/2}} \end{pmatrix} \tag{28}$$

is an element of the Hermitian braided matrices $BM_q(2)$ with zero q-determinant. The creation operator $b^\dagger$, the annihilation operator b and the number operator N satisfy the oscillator relation

$$bb^\dagger - q^2 b^\dagger b = q^{4N}. \tag{29}$$

One of the main motivations for introducing a deformation parameter q into physics is the possibility of regularizing infinities. It is possible that $q \neq 1$ will give quantum corrections on the Planck scale. Then, after renormalization one can can set $q = 1$ to obtain the effective theory at low energies. The importance and many uses of the $SU(2)$ group in present-day physics makes it crucial to investigate all properties of its q-deformations. We have shown that the matrix elements of Hermitian braided group matrices are q-deformed generators of $SU(2)$.

The Schwinger-type construction of these generators which we have accomplished can be applied to other braided matrices. Probably all Lie algebras can be generalized in this way and can be shown to possess the interesting features we have investigated in this letter.

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